

COXETER POLYNOMIALS OF SALEM TREES

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ABSTRACT. We compute the Coxeter polynomial of a family of Salem trees and also the limit of the spectral radius of their Coxeter transformations as the number of their vertices tends to infinity. We also prove that if z is a root of multiplicities m_1, \dots, m_k for the Coxeter polynomials of the trees $\mathcal{T}_1, \dots, \mathcal{T}_k$, then z is a root for the Coxeter polynomial of their join, of multiplicity at least $\min\{m - m_1, \dots, m - m_k\}$ where $m = m_1 + \dots + m_k$.

1. INTRODUCTION AND PRELIMINARIES

In [14], Lakatos determines the limit of the spectral radii of the Coxeter transformations of particular infinite sequences of starlike trees. In the present paper we generalize the result of Lakatos [14] to a wider range of trees. In addition, our idea of proof is different from the one in [14].

We use the same terminology as in [14, 24] and [27]. We denote by $\mathbb{N} \subseteq \mathbb{Z}$ the set of nonnegative integers and the ring of integers, respectively. The algebra of the $n \times n$ square integer matrices is denoted by $\mathbb{M}_n(\mathbb{Z})$, where $n \in \mathbb{N}$. We consider only simple graphs (i.e., graphs without multiple edges and loops), $\Gamma = (\Gamma_0, \Gamma_1)$ with the set of vertices $\Gamma_0 = \{v_1, \dots, v_n\}$ and Γ_1 the set of edges, where $(v_i, v_j) \in \Gamma_1$ if there is an edge connecting the vertices v_i and v_j .

Assume that $\Gamma = (\Gamma_0, \Gamma_1)$ is a simple graph with the set of enumerated vertices $\Gamma_0 = \{v_1, \dots, v_n\}$. We recall that the **adjacency matrix** of the graph Γ is the $n \times n$ symmetric matrix

$$(1.1) \quad \text{Ad}_\Gamma = [a_{ij}] \in \mathbb{M}_n(\mathbb{Z})$$

with $a_{ij} = 1$, if $(v_i, v_j) \in \Gamma_1$ and $a_{ij} = 0$, otherwise. The **characteristic polynomial** of Γ is defined to be the polynomial

$$(1.2) \quad \chi_\Gamma(t) := \det(t \cdot I_n - \text{Ad}_\Gamma) \in \mathbb{Z}[t]$$

where $I_n = [\delta_{ij}]$ is the identity matrix in $\mathbb{M}_n(\mathbb{Z})$. It is clear that $\chi_\Gamma(t)$ does not depend on the enumeration v_1, \dots, v_n of the vertices in Γ_0 , see [4] and [6].

Let \mathbb{R}^n be the standard n dimensional real vector space with the standard basis e_1, \dots, e_n . Given $i \in \{1, \dots, n\}$, the i th reflection of Γ is defined to be

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the \mathbb{R} -linear automorphism $\sigma_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by the formula

$$(1.3) \quad \sigma_i(e_j) = e_j - (2\delta_{ij} - a_{ij})e_i.$$

The subgroup W_Γ of the general linear group $GL(\mathbb{R}^n) \cong GL(n, \mathbb{R})$ generated by the reflections $\sigma_1, \dots, \sigma_n$ of Γ is called the **Weyl group** of Γ and has the presentation

$$(1.4) \quad W_\Gamma = \langle \sigma_1, \sigma_2, \dots, \sigma_n : (\sigma_i \sigma_j)^{m_{ij}} = 1 \rangle$$

where $M = [m_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ is the matrix defined by $m_{ii} = 1$ for all $i = 1, \dots, n$ and $m_{ij} = a_{ij} + 2$ for all $i \neq j$, see [3, 11, 30]. The product $\Phi_\Gamma = \sigma_1 \cdot \dots \cdot \sigma_n \in W_\Gamma$ is defined to be the **Coxeter transformation** of the graph Γ , see [17]. Obviously, it depends on the enumeration of the vertices v_1, \dots, v_n of Γ , see 1.1 for details. We recall that the Coxeter transformations were first studied by Coxeter in [5] where he showed that their eigenvalues have remarkable properties, see also Bourbaki [3] and Humphreys [11].

Throughout this paper, we assume that Γ is a tree $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ with enumerated vertices $\mathcal{T}_0 = \{v_1, \dots, v_n\}$, $\text{Ad}_\mathcal{T} = [a_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ is its adjacency matrix, and

$$(1.5) \quad \Phi_\mathcal{T} = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_n \in W_\mathcal{T}$$

is its Coxeter transformation, with respect to the enumeration v_1, v_2, \dots, v_n . The Coxeter polynomial of the tree \mathcal{T} is defined to be the characteristic polynomial of $\Phi_\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that is, the polynomial (see [11, 17, 25])

$$(1.6) \quad \text{cox}_\mathcal{T}(t) := \det(t \cdot \text{id}_{\mathbb{R}^n} - \Phi_\mathcal{T}) \in \mathbb{Z}[t].$$

Since \mathcal{T} is a tree, the characteristic polynomial of the transformation $\Phi_\mathcal{T}$ does not depend on the enumeration of the vertices v_1, \dots, v_n . Indeed, if $v_{\epsilon(1)}, \dots, v_{\epsilon(n)}$ is obtained from v_1, \dots, v_n by a permutation $\epsilon \in S_n$ then the Coxeter transformation $\Phi_\mathcal{T}^\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$ corresponding to the enumeration $v_{\epsilon(1)}, \dots, v_{\epsilon(n)}$ is conjugate with $\Phi_\mathcal{T}$, see [25, Proposition 2.2], [11, Proposition 3.16], [3, 17] and the following remark for details.

Remark 1.1. (a) The Coxeter polynomial $\text{cox}_\Delta(t)$ is also defined and studied in [24, 25] and [26] in a more general setting of loop-free edge-bipartite multigraphs $\Delta = (\Delta_0, \Delta_1 = \Delta_1^- \cup \Delta_1^+)$, with $\Delta_0 = \{v_1, v_2, \dots, v_n\}$ and a separated bipartition $\Delta_1 = \Delta_1^- \cup \Delta_1^+$ of the set of edges. The class of loop-free edge-bipartite multigraphs contains all simple graphs, loop-free multigraphs, and simple signed graphs, see [32].

The definition of $\text{cox}_\Delta(t) \in \mathbb{Z}[t]$ for an edge-bipartite multigraph Δ , differs from the one given in (1.6) for simple graphs, and depends on the upper triangular Gram matrix $\check{G}_\Delta = [d_{ij}^\Delta] \in GL(n, \mathbb{Z})$ where $d_{ij}^\Delta = 1$ for $i = j$, d_{ij}^Δ is the number of edges between v_i and v_j , with $i < j$, lying in Δ_1^+ and $-d_{ij}^\Delta$ is the number of edges between v_i and v_j , with $i < j$, lying in Δ_1^- .

In [24, 25] and [26], with any loop-free edge-bipartite multigraph $\Delta = (\Delta_0, \Delta_1 = \Delta_1^- \cup \Delta_1^+)$ the Coxeter matrix $\text{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-\text{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is associated and its characteristic polynomial

$$(1.7) \quad \text{cox}_\Delta(t) := \det(t \cdot I_n - \text{Cox}_\Delta) \in \mathbb{Z}[t],$$

called the **Coxeter polynomial** of Δ is self-reciprocal in the sense that $\text{cox}_\Delta(t) = t^n \text{cox}_\Delta(\frac{1}{t})$, see Lemma 2.8 (c3)-(c4) in [23]. The **Coxeter transformation** of Δ is defined to be the group automorphism

$$(1.8) \quad \Phi_\Delta : \mathbb{Z}^n \rightarrow \mathbb{Z}^n, v \mapsto v \cdot \text{Cox}_\Delta.$$

It is proved in [25, Proposition 2.2] that in the case when the underlying multigraph $\overline{\Delta}$ of Δ is a tree, the Coxeter polynomial does not depend on the enumeration of the vertices v_1, \dots, v_n . Hence, in view of the sink-source reflection technique applied in [1, Proposition VII.4.7], the Coxeter polynomial $\text{cox}_\Delta(t)$ (1.7) of Δ coincides with the Coxeter polynomial $\text{cox}_{\overline{\Delta}}(t)$ of the tree $\mathcal{T} = \overline{\Delta}$ (in the sense of (1.6)).

The reader is also referred to the recent papers [12, 13], where the irreducible and reduced root systems in the sense of Bourbaki [3] are studied in connection with roots of positive connected edge-bipartite graphs.

(b) The Coxeter polynomial is also defined in [22] and [27], for any finite poset $J \equiv (J, \preceq)$, with $J = \{1, \dots, n\}$, as

$$(1.9) \quad \text{cox}_J(t) := \det(t \cdot I_n - \text{Cox}_J) \in \mathbb{Z}[t]$$

where $\text{Cox}_J = -C_J \cdot C_J^{-\text{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is the Coxeter matrix of J and $C_J := [c_{ij}] \in \mathbb{M}(\mathbb{Z})$ is its incidence matrix, with $c_{ij} = 1$, for $i \preceq j$, and $c_{ij} = 0$ if $i \not\preceq j$. It is shown that if the Hasse diagram $H := \mathcal{H}_J$ of J is a tree, then the Coxeter polynomial $\text{cox}_J(t)$ (1.9) of J coincides with the Coxeter polynomial $\text{cox}_H(t)$ of the tree $\mathcal{T} = H$ (in the sense of (1.6)).

By applying 1.1(a) we get the following useful fact

Corollary 1.2. *Assume that $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ is a tree with enumerated vertices v_1, \dots, v_n and let $\check{G}_\mathcal{T} = [d_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ be the upper triangular Gram matrix of \mathcal{T} , with $d_{11} = \dots = d_{nn} = 1$, $d_{ij} = -1$ if $i < j$ and there is an edge (v_i, v_j) in \mathcal{T}_1 and $[d_{ij}] = 0$, otherwise.*

(a) *The Coxeter transformation $\Phi_\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (1.5) of the tree \mathcal{T} restricts to the group automorphism $\Phi_\mathcal{T} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ defined by the formula*

$$\Phi_\mathcal{T}(u) = u \cdot \text{Cox}_\mathcal{T}$$

where $\text{Cox}_\mathcal{T} := -\check{G}_\mathcal{T} \cdot \check{G}_\mathcal{T}^{-\text{tr}} \in \mathbb{M}_n(\mathbb{Z})$ is the Coxeter matrix of \mathcal{T} viewed as an edge-bipartite graph, with \mathcal{T}_1^+ empty.

(b) *The Coxeter polynomial $\text{cox}_\mathcal{T}(t)$ (1.6) of the tree \mathcal{T} coincides with the Coxeter polynomial $\text{cox}_\mathcal{T}(t) = \det(t \cdot I_n - \text{Cox}_\Delta)$ (1.7) of \mathcal{T} viewed as an edge-bipartite tree.*

(c) *The Coxeter polynomial $\text{cox}_\mathcal{T}(t)$ (1.6) of the tree \mathcal{T} is self-reciprocal and does not depend on the enumeration of the vertices v_1, \dots, v_n of the tree \mathcal{T} .*

Proof. We view \mathcal{T} as an edge-bipartite graph, with $\mathcal{T}_1 = \mathcal{T}_1^- \cup \mathcal{T}_1^+$ where \mathcal{T}_1^+ is the empty set. Then the matrix $\check{G} = [d_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ coincides with the upper triangular Gram matrix $\check{G}_\Delta = [a_{ij}^\Delta]$ defined in 1.1(a). Then the corollary is a consequence of 1.1(a). \square

The most important families of trees are the trees of type *ADE* given in Figure 1. These trees are known as the simply laced Dynkin diagrams. There is a long list of objects which admit an *ADE* classification, meaning that there is an equivalence between equivalence classes of objects of the given type and the *ADE* graphs (see for example [9]). Examples of these objects include the

- simply laced finite Coxeter groups,
- simply laced simple Lie algebras,
- platonic solids,
- quivers of finite representation types,
- Kleinian singularities,
- finite subgroups of the $SU(2)$ group.

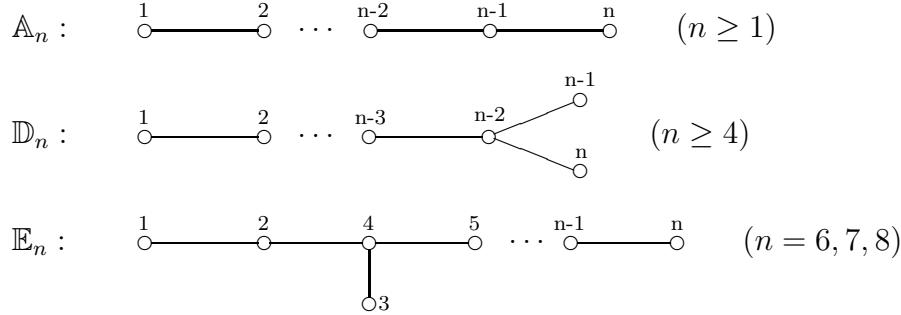


FIGURE 1. Simply laced Dynkin diagrams

Note that the graphs \mathbb{E}_n are defined in general for all $n \geq 3$, where $\mathbb{E}_3 = \mathbb{A}_2 \oplus \mathbb{A}_1$ and for $n \geq 4$ are defined as shown in Figure 1. The graphs \mathbb{E}_n were studied extensively in [8] where their Coxeter polynomials were completely factored into cyclotomic and Salem polynomials. The Coxeter polynomials of the *ADE* graphs are well known and have been calculated many times (see for instance [2, 3, 7, 8, 25, 27, 30]). One of the main aims of this paper is to find a universal formula for the Coxeter polynomials of a family of trees which we denote by $S_{p_1, \dots, p_k}^{(i)}$. For specific values of $i, k, p_1, \dots, p_k \in \mathbb{N}$ we obtain the *ADE* graphs.

To define the trees $S_{p_1, \dots, p_k}^{(i)}$, we recall that the join of the simple graphs $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, with a fixed vertex v_i in each of the graphs Γ_i , is the graph obtained by adding a new vertex and joining that to v_i for all $i = 1, 2, \dots, k$ (see [30]).

For $k \in \mathbb{N}, p_1, \dots, p_k \in \mathbb{N}$ and $i \in \{0, 1, 2, \dots, k\}$, we define the tree $S_{p_1, \dots, p_k}^{(i)}$ to be the join of the Dynkin diagrams $\mathbb{D}_{p_1}, \dots, \mathbb{D}_{p_i}$ and $\mathbb{A}_{p_{i+1}}, \dots, \mathbb{A}_{p_k}$, on their vertices numbered 1, as shown in Figure 3.

The trees $S_{p_1, \dots, p_k}^{(0)}$ are the stars $\mathbb{T}_{p_1-1, \dots, p_k-1}$ defined in [20], which are the join of the Dynkin diagrams $\mathbb{A}_{p_1-1}, \dots, \mathbb{A}_{p_k-1}$. These are also the wild stars defined in [14].

To the best of my knowledge the graphs $S_{p_1, \dots, p_k}^{(i)}$ for $i \geq 1$ are defined here for the first time. For particular values of i and p_j , we get some well-known

trees. For example, for $k = 2, i = 0, p_1 = 1, p_2 = n - 2$ we obtain the Dynkin diagrams \mathbb{A}_n , for $k = 3, i = 0, p_1 = 1, p_2 = 1, p_3 = n - 3$ we obtain the Dynkin diagrams \mathbb{D}_n , for $k = 3, i = 0, p_1 = 1, p_2 = 2, p_3 = n - 4$ we obtain the diagrams \mathbb{E}_n and for $k = 3, i = 1, p_1 = n - 2, p_2 = p_3 = 1$ we obtain the Euclidean Dynkin diagrams $\tilde{\mathbb{D}}_n$ (see Figure 2). Note that $S_{1,2,6}^{(0)} = \mathbb{E}_{10}$ and the Coxeter polynomial $\text{cox}_{\mathbb{E}_{10}}(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$ is the well known Lehmer's polynomial which is conjectured to have the smallest Mahler measure among the monic integer non-cyclotomic polynomials (see [29]).

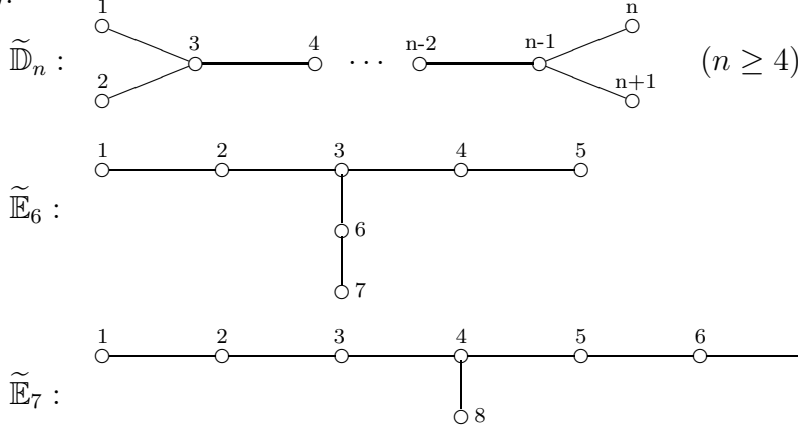


FIGURE 2. The Euclidean diagrams $\tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6$ and $\tilde{\mathbb{E}}_7$

Let $p(t)$ be a monic polynomial with integer coefficients. We denote the set of its roots $\{z \in \mathbb{C} : p(z) = 0\}$ by $Z(p(t))$ and the maximum value of the set $\{|z| : z \in Z(p)\}$ by $\rho(p(t))$. For example, for the polynomials $\text{cox}_{\mathbb{A}_n}(t), \text{cox}_{\mathbb{D}_n}(t)$ we have $\rho(\text{cox}_{\mathbb{A}_n}(t)) = \rho(\text{cox}_{\mathbb{D}_n}(t)) = 1$ while for the polynomials $\text{cox}_{\mathbb{E}_n}(t)$ for $n \geq 10$ we have $\rho(\text{cox}_{\mathbb{E}_n}(t)) > 1$ (see [8] and [15]).

Assuming that the polynomial $p(t)$ is irreducible then, if all of its roots lie on the unit circle (or equivalently $\rho(p(t)) = 1$), it is called a *cyclotomic polynomial*.

Assuming now that the polynomial $p(t)$ is irreducible, non-cyclotomic with only one root outside the unit circle then, if it has at least one root on the unit circle it is called a *Salem polynomial* while if it has no roots on the unit circle it is called a *Pisot polynomial* (see [15]).

It is not hard to see that cyclotomic and Salem polynomials are self-reciprocal. This follows from the following facts. The polynomial $p(t)$ of degree n is irreducible if and only if the polynomial $p^*(t) := t^n p(\frac{1}{t})$, which we call the reciprocal of $p(t)$, is irreducible. If α lies on the unit circle then α is a root of $p(t)$ if and only if $\frac{1}{\alpha}$ is also a root of $p(t)$.

We recall from [15] the following definition.

Definition 1.3. (a) A tree \mathcal{T} is said to be *cyclotomic* if all roots of the Coxeter polynomial $\text{cox}_{\mathcal{T}}(t)$ are on the unit disk or equivalently $\text{cox}_{\mathcal{T}}(t)$ is a product of cyclotomic polynomials.

(b) A tree \mathcal{T} is called a *Salem tree* if the Coxeter polynomial $\text{cox}_{\mathcal{T}}(t)$ has only one root outside the unit circle or equivalently $\text{cox}_{\mathcal{T}}(t)$ is a product of one of the Salem polynomials and some cyclotomic polynomials.

2. MAIN RESULTS

In this paper we are mainly concerned with the case $k = 3$ (i.e. with the trees $S_{p,q,r}^{(i)}$) and prove four theorems about the Coxeter polynomials $\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t)$. In Theorem 2.1 we present a recursive relation for the Coxeter polynomials of the trees $S_{p_1, \dots, p_k}^{(i)}$ and we use it in Theorem 2.2 to find the Coxeter polynomials of the trees $S_{p,q,r}^{(i)}$ for all $i = 0, 1, 2, 3$. In Theorem 2.3 we show that the limits $\lim_{p \rightarrow \infty} \rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right)$, $\lim_{q \rightarrow \infty} \rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right)$ and $\lim_{r \rightarrow \infty} \rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right)$ are Pisot numbers. We also show that

$$\lim_{p,q,r \rightarrow \infty} \rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right) = 2, \text{ for all } i = 0, 1, 2, 3.$$

It was shown by Lakatos [14] that

$$\lim_{p_1, \dots, p_k \rightarrow \infty} \rho \left(\text{cox}_{S_{p_1, \dots, p_k}^{(0)}}(t) \right) = k - 1, \text{ for } k \in \mathbb{N}.$$

In Theorem 2.4 we generalize that result by showing that

$$\lim_{p_1, \dots, p_k \rightarrow \infty} \rho \left(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) \right) = k - 1, \text{ for all } i \in \{0, 1, \dots, k\}.$$

We mention here that the multiple limits $\lim_{p_1, \dots, p_i \rightarrow \infty} \alpha_n$ are the iterated limits $\lim_{p_1 \rightarrow \infty} (\dots (\lim_{p_i \rightarrow \infty} \alpha_n))$.

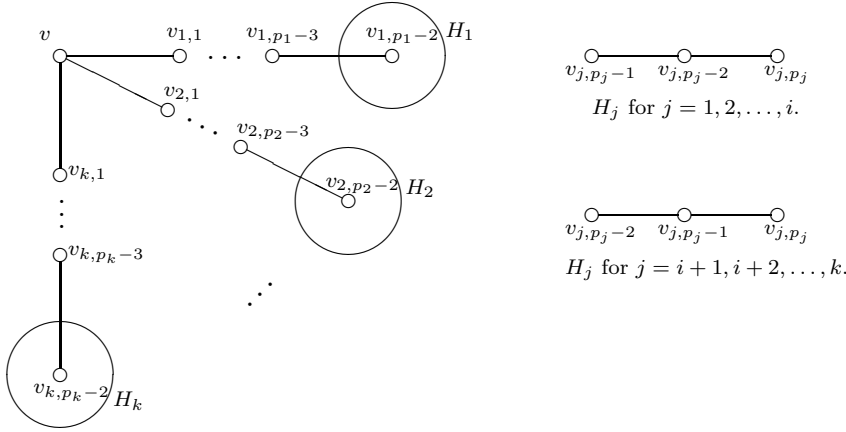


FIGURE 3. The trees $S_{p_1, \dots, p_k}^{(i)}$

Theorem 2.1. *Let $k, p_1, \dots, p_k \in \mathbb{N}$ and $p_1 \geq 2$. Then*

$$\text{cox}_{S_{p_1, \dots, p_k}^{(0)}}(t) = (t+1) \text{cox}_{S_{p_1-1, \dots, p_k}^{(0)}}(t) - t \text{cox}_{S_{p_1-2, \dots, p_k}^{(0)}}(t).$$

If $k \geq 2$ and $p_1 \geq 3$ then

$$\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) = (t+1) \left[\text{cox}_{S_{p_2, \dots, p_k, p_1-1}^{(i-1)}}(t) - t \text{cox}_{S_{p_2, \dots, p_k, p_1-3}^{(i-1)}}(t) \right],$$

for all $i \in \{1, \dots, k\}$

Theorem 2.2. (a) For $i \leq 2$, the Coxeter polynomial $\text{cox}_{S_{p,q,r}^{(i)}}(t)$ of the tree $S_{p,q,r}^{(i)}$, is given by the formula

$$\text{cox}_{S_{p,q,r}^{(i)}}(t) = \frac{(t+1)^i}{t-1} \left[t^{r+2} F_{p,q}^{(i)}(t) - (F_{p,q}^{(i)})^*(t) \right],$$

where

$$\begin{aligned} F_{p,q}^{(0)}(t) &= t^{p+q} - \text{cox}_{\mathbb{A}_{p-1}}(t) \text{cox}_{\mathbb{A}_{q-1}}(t), \\ F_{p,q}^{(1)}(t) &= t^{p+q-2}(t-1) - (t^{p-2}+1) \text{cox}_{\mathbb{A}_{q-1}}(t) \text{ and} \\ F_{p,q}^{(2)}(t) &= t^{p+q-4}(t-1)^2 - (t^{p-2}+1)(t^{q-2}+1). \end{aligned}$$

(b) The Coxeter polynomial $\text{cox}_{S_{p,q,r}^{(3)}}(t)$ is given by the formula

$$\text{cox}_{S_{p,q,r}^{(3)}}(t) = (t+1)^3 \left[t^r F_{p,q}^{(3)}(t) + (F_{p,q}^{(3)})^*(t) \right],$$

where $F_{p,q}^{(3)}(t) = F_{p,q}^{(2)}(t)$.

Theorem 2.3. Let $\rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right)$ be the spectral radius of the Coxeter transformation of $S_{p,q,r}^{(i)}$. Then we have

- (1) $\lim_{r \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right) = \rho\left(F_{p,q}^{(i)}(t)\right)$ and $\rho\left(F_{p,q}^{(i)}(t)\right)$ is a Pisot number for $i = 0, 1, 2$,
- (2) $\lim_{p \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right) = \rho\left(F_{q,r}^{(i-1)}(t)\right)$ for $i = 1, 2, 3$,
- (3) $\lim_{p,q \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right) = \rho\left(t^{r+2} - 2t^{r+1} + 1\right)$ for $i = 0, 1, 2$,
- (4) $\lim_{q,r \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right) = \rho\left(t^p - 2t^{p-1} - 1\right)$ for $i = 1, 2, 3$ and
- (5) $\lim_{p,q,r \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right) = 2$ for all $i = 0, 1, 2, 3$.

Theorem 2.4. For $k, p_1, \dots, p_k \in \mathbb{N}$ and all $i \in \{0, 1, \dots, k\}$ we have

$$\lim_{p_1, \dots, p_k \rightarrow \infty} \rho\left(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t)\right) = k - 1.$$

Remark 2.5. (a) Note that for $i = 1$ or $i = 3$ the trees $S_{p,q,r}^{(i)}$ and $S_{r,q,p}^{(i)}$ are the same and therefore the case $i = 3$ in (1) of Theorem 2.3 is given in (2). Similarly the limit $\lim_{p \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(0)}}(t)\right)$ can be found using the result of (1). The same holds for the cases of (3) and (4); the double limit $\lim_{p,q \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(3)}}(t)\right)$ is obtained from (4) and $\lim_{q,r \rightarrow \infty} \rho\left(\text{cox}_{S_{p,q,r}^{(i)}}(t)\right)$ from (3).

(b) In [15] it was shown by James McKee and Chris Smyth that if a noncyclotomic tree is the join of cyclotomic trees then it is a Salem tree. The cyclotomic trees were classified in [28]; they are the subgraphs of the Euclidean diagram $\tilde{\mathbb{E}}_8 = \mathbb{E}_9$ and of the Euclidean diagrams of Figure 2 (see also [15, 19]). In [15] the Salem trees were classified and they include the joins of cyclotomic trees which are not cyclotomic. It follows from this classification that the cyclotomic cases of the trees $S_{p_1, \dots, p_k}^{(i)}$ are those for $k =$

$i = 2$ or $k = 3, i = 0, p_1 = p_2 = p_3 = 2$ or $k = 3, i = 0, p_1 = 1, p_2 = p_3 = 3$ or $k = 3, i = 0, p_1 = 1, p_2 = 2, p_3 = 5$ and subgraphs of these. For all the other cases, $S_{p_1, \dots, p_k}^{(i)}$ are Salem trees.

(c) We recall that the Mahler measure of a monic integer polynomial $f(t)$ is $M(f) = \prod \{|z| : z \in Z(f(t)), |z| \geq 1\}$ (see [29]). We can easily see that if f is cyclotomic, Salem or Pisot then its Mahler measure is $M(f) = \rho(f(t))$. Lehmer's problem asks if we can chose f with Mahler measure arbitrarily close to 1. Since the polynomials $\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t)$ have at most one root outside the unit circle it follows that their Mahler measure is $\rho(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t))$. Theorem 2.2 in connection with 3.3 can be used to verify Lehmer's conjecture for the family of the polynomials $\text{cox}_{S_{p,q,r}^{(i)}}(t)$, asserting that the smallest Mahler measure, larger than 1, is the Mahler measure of the polynomial $\text{cox}_{S_{1,2,6}^{(0)}}(t) = \text{cox}_{\mathbb{E}_{10}}(t)$ (see also [15] and the recent papers [16, 18]).

Example 2.6. For the case of the Dynkin diagrams \mathbb{D}_n , Theorem 2.2 gives

$$\begin{aligned} \text{cox}_{\mathbb{D}_n}(t) &= \text{cox}_{S_{1,1,n-3}^{(0)}}(t) \\ &= \frac{1}{t-1} (t^{n-1}(t^2-1) + t^2-1) = t^n + t^{n-1} + t + 1. \end{aligned}$$

For the Euclidean diagrams $\widetilde{\mathbb{D}}_n$, Theorem 2.2 gives

$$\begin{aligned} \text{cox}_{\widetilde{\mathbb{D}}_n}(t) &= \text{cox}_{S_{n-2,1,1}^{(1)}}(t) \\ &= \frac{t+1}{t-1} [t^3(t^{n-2} - t^{n-3} - t^{n-4} - 1) + t^{n-2} + t^2 + t - 1] \\ &= (t^{n-2} - 1)(t-1)(t+1)^2 \end{aligned}$$

and for the diagrams \mathbb{E}_n it gives

$$\text{cox}_{\mathbb{E}_n}(t) = \text{cox}_{S_{1,2,n-4}^{(0)}}(t) = \frac{1}{t-1} [t^{n-2}(t^3 - t - 1) + t^3 + t^2 - 1].$$

All these agree with the known formulas of the Coxeter polynomials of the diagrams $\mathbb{D}_n, \widetilde{\mathbb{D}}_n$ and \mathbb{E}_n (see [7, 8] and [25, Proposition 2.3]).

We also prove the following theorem concerning joins of trees.

Theorem 2.7. *Let \mathcal{T} be the join of the trees $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(k)}$, $k \geq 2$. Suppose that z is a root of the polynomial $\text{cox}_{\mathcal{T}^{(i)}}(t)$ with multiplicity m_i . Then z is also a root of the polynomial $\text{cox}_{\mathcal{T}}(t)$ with multiplicity at least*

$$\min\{m - m_i : i = 1, 2, \dots, k\}$$

where $m = m_1 + m_2 + \dots + m_k$.

Remark 2.8. (a) According to [31] if the common root z of the polynomials $\text{cox}_{\mathcal{T}_1}(t), \dots, \text{cox}_{\mathcal{T}_k}(t)$ is $z \neq \pm 1$ then its multiplicity m_i is 1. Therefore in that case Theorem 2.7 gives that z is a root of $\text{cox}_{\mathcal{T}}(t)$ with multiplicity at least $k-1$. This result was proved in [8, Theorem 3.1]. For $z = \pm 1$ however, z can be a root of $\text{cox}_{\mathcal{T}}(t)$ with multiplicity less than $k-1$. For example, consider the join \mathcal{T} of the Euclidean diagrams $\widetilde{\mathbb{D}}_4$ as shown in Figure 4. The polynomials $\text{cox}_{\mathcal{T}}(t)$ and $\text{cox}_{\widetilde{\mathbb{D}}_4}(t)$ both have 1 as a root with multiplicity 2.

(b) Now suppose that \mathcal{T} is the join of the trees $\mathcal{T}_1, \mathcal{T}_2$ and z is the common root of the Coxeter polynomials $\text{cox}_{\mathcal{T}_1}(t)$ and $\text{cox}_{\mathcal{T}_2}(t)$. Then Theorem 2.7 generalizes a theorem due to Kolmykov [30] (see also [8, Theorem 1.5]) asserting that z is a root of the Coxeter polynomial $\text{cox}_{\mathcal{T}}(t)$.

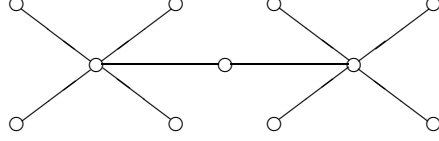


FIGURE 4. The join of two \mathbb{D}_4 diagrams

For the convenience of the reader we include all theorems that will be used, in several cases with proofs, thus making this paper self-contained. This is done in Section 3. In Section 4 we prove Theorems 2.1, 2.2, 2.3, 2.4 and 2.7 formulated in Section 2.

3. GENERALITIES ON COXETER POLYNOMIALS

In this section we collect and prove some results that we need in the proof of Theorems 2.2 2.3 2.4 and 2.7.

The following proposition is due to Subbotin and Sumin and the proof we present here is taken from [30].

Proposition 3.1. *Assume that $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ is a tree and let $e = (v_1, v_2) \in \mathcal{T}_1$ be a splitting edge of the tree \mathcal{T} that splits it to the trees $\mathcal{R} = (\mathcal{R}_0, \mathcal{R}_1)$ and $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$. Assume that $v_1 \in \mathcal{R}_0$ and $v_2 \in \mathcal{S}_0$. Then*

$$\text{cox}_{\mathcal{T}}(t) = \text{cox}_{\mathcal{R}}(t) \text{cox}_{\mathcal{S}}(t) - t \text{cox}_{\tilde{\mathcal{R}}}(t) \text{cox}_{\tilde{\mathcal{S}}}(t)$$

where $\tilde{\mathcal{R}} = (\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_1)$, $\tilde{\mathcal{S}} = (\tilde{\mathcal{S}}_0, \tilde{\mathcal{S}}_1)$ are the subgraphs of \mathcal{R}, \mathcal{S} with the vertex sets $\tilde{\mathcal{R}}_0 = \mathcal{R}_0 \setminus \{v_1\}$ and $\tilde{\mathcal{S}}_0 = \mathcal{S}_0 \setminus \{v_2\}$.

Proof. We enumerate the vertices of \mathcal{R} and \mathcal{S} as $\mathcal{R}_0 = \{u_1, u_2, \dots, u_k\}$ and $\mathcal{S}_0 = \{u_{k+1}, u_{k+2}, \dots, u_{k+m}\}$, where $v_1 = u_k$ and $v_2 = u_{k+1}$. Let $\hat{e} = \{e_1, \dots, e_{k+m}\}$ be the standard basis for the vector space \mathbb{R}^{k+m} , and let V_1 be the vector subspace of \mathbb{R}^{k+m} with basis $\hat{e}_1 = \{e_1, e_2, \dots, e_k\}$ and V_2 the vector subspace of \mathbb{R}^{k+m} with basis $\hat{e}_2 = \{e_{k+1}, e_{k+2}, \dots, e_{k+m}\}$. Also let σ_i be the i th reflection of \mathcal{T} . Then $\Phi_{\mathcal{R}} = \sigma_1 \sigma_2 \dots \sigma_k$ is a Coxeter transformation of \mathcal{R} , $\Phi_{\mathcal{S}} = \sigma_{k+1} \sigma_{k+2} \dots \sigma_{k+m}$ is a Coxeter transformation of \mathcal{S} and $\Phi_{\mathcal{T}} = \Phi_{\mathcal{R}} \Phi_{\mathcal{S}}$ is a Coxeter transformation of \mathcal{T} . If R, S are the matrices corresponding to $\Phi_{\mathcal{R}}, \Phi_{\mathcal{S}}$ with respect to the bases \hat{e}_1, \hat{e}_2 then with respect to the basis \hat{e} the Coxeter transformation $\Phi_{\mathcal{T}}$ corresponds to the matrix

$$\begin{pmatrix} R & E_{k1} \\ 0_{mk} & I_m \end{pmatrix} \cdot \begin{pmatrix} I_k & 0_{km} \\ E_{1k} & S \end{pmatrix},$$

where E_{ij} is the matrix with all entries zero except the i, j entry which is 1 and 0_{ij} is the $i \times j$ zero matrix. The Coxeter polynomial of \mathcal{T} is then given

by

$$\text{cox}_{\mathcal{T}}(t) = \det(tI_{k+m} - \Phi_{\mathcal{T}}) = \det \begin{pmatrix} tI_k - R - E_{k,k} & -E_{k,1}S \\ -E_{1,k} & tI_m - S \end{pmatrix}.$$

Subtracting the $k+1^{\text{th}}$ row from the k^{th} row we obtain

$$\text{cox}_{\mathcal{T}}(t) = \det \begin{pmatrix} tI_k - R & -tE_{k,1} \\ -E_{1,k} & tI_m - S \end{pmatrix}.$$

Expanding the determinant with respect to the k^{th} row we deduce that

$$\text{cox}_{\mathcal{T}}(t) = \text{cox}_{\mathcal{R}}(t) \text{cox}_{\mathcal{S}}(t) - t \text{cox}_{\widehat{\mathcal{R}}}(t) \text{cox}_{\widehat{\mathcal{S}}}(t).$$

□

The following well-known lemma says that the eigenvalues of a bipartite graph are symmetric around 0, see [4, 6].

Lemma 3.2. *Let Γ be a bipartite graph. If λ is an eigenvalue of the adjacency matrix Ad_{Γ} of Γ then $-\lambda$ is an eigenvalue of Ad_{Γ} .*

Proof. Enumerate the vertices of Γ such that its adjacency matrix has the form

$$\text{Ad}_{\Gamma} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Suppose that $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of Ad_{Γ} with eigenvalue λ . Then $\begin{pmatrix} -x \\ y \end{pmatrix}$ is an eigenvector of Ad_{Γ} with eigenvalue $-\lambda$. □

The next lemma is due to Hoffman and Smith (see [10]).

Lemma 3.3. *If $k, p_1, \dots, p_k \in \mathbb{N}$, $0 \leq i \leq k$ and $p_j < p'_j$, for some $1 \leq j \leq k$, then*

$$\begin{aligned} (1) \quad & \rho \left(\text{cox}_{S_{p_1, \dots, p_j, \dots, p_k}^{(i)}}(t) \right) \leq \rho \left(\text{cox}_{S_{p_1, \dots, p'_j, \dots, p_k}^{(i)}}(t) \right) \text{ if } j > i \text{ and} \\ (2) \quad & \rho \left(\text{cox}_{S_{p_1, \dots, p_j, \dots, p_k}^{(i)}}(t) \right) \geq \rho \left(\text{cox}_{S_{p_1, \dots, p'_j, \dots, p_k}^{(i)}}(t) \right) \text{ if } j \leq i. \end{aligned}$$

Moreover, the equalities hold if and only if the tree $S_{p_1, \dots, p'_j, \dots, p_k}^{(i)}$ is cyclotomic.

We will also need the following lemma.

Lemma 3.4. *Suppose that $f_n(t) = t^n g(t) + h(t)$ is a sequence of functions such that g, h are continuous, $f_n(z_n) = 0$ for all $n \in \mathbb{N}$ and that $\lim_{n \rightarrow \infty} z_n = z_0$. If $|z_0| > 1$ then $g(z_0) = 0$ while if $|z_0| < 1$ then $h(z_0) = 0$.*

Proof. Suppose that $|z_0| > 1$. The function h is continuous and $|g(z_n)| = \frac{|h(z_n)|}{|z_n^n|}$. Therefore $\lim_{n \rightarrow \infty} |g(z_n)| = 0$. Since $|g(z_0)| - |g(z_n)| \leq |g(z_0) - g(z_n)| \xrightarrow{n \rightarrow \infty} 0$, we conclude that $g(z_0) = 0$. The proof for the case $|z_0| < 1$ is similar. □

4. PROOF OF MAIN THEOREMS

In this section we prove Theorems 2.1, 2.2, 2.3, 2.4 and 2.7.

Proof of Theorem 2.1. For $p_1 \geq 2$ we split the tree $S_{p_1, \dots, p_k}^{(0)}$ by removing the edge $(v_{1, p_1-1}, v_{1, p_1})$ and we apply Proposition 3.1 to get

$$\begin{aligned} \text{cox}_{S_{p_1, \dots, p_k}^{(0)}}(t) &= \text{cox}_{\mathbb{A}_1}(t) \text{cox}_{S_{p_1-1, \dots, p_k}^{(0)}}(t) - t \text{cox}_{S_{p_1-2, \dots, p_k}^{(0)}}(t) \\ &= (t+1) \text{cox}_{S_{p_1-1, \dots, p_k}^{(0)}}(t) - t \text{cox}_{S_{p_1-2, \dots, p_k}^{(0)}}(t). \end{aligned}$$

We used that $\text{cox}_{\mathbb{A}_1}(t) = t+1$ which can be easily verified from the definition of the Coxeter polynomial.

For $k \geq 2, p_1 \geq 3$ and $1 \leq i \leq k$ if split the tree $S_{p_1, \dots, p_k}^{(0)}$ by removing the edge $(v_{1, p_1-2}, v_{1, p_1})$ we end up with \mathbb{A}_1 and the join of $i-1$ Dynkin diagrams of type $\mathbb{D}_{p_2}, \dots, \mathbb{D}_{p_i}$ and $k-i+1$ Dynkin diagrams of type $\mathbb{A}_{p_{i+1}}, \dots, \mathbb{A}_{p_k}, \mathbb{A}_{p_1-1}$. We apply Proposition 3.1 to the edge $(v_{1, p_1-2}, v_{1, p_1})$ to get

$$\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) = \text{cox}_{\mathbb{A}_1}(t) \left[\text{cox}_{S_{p_2, \dots, p_k, p_1-1}^{(i-1)}}(t) - t \text{cox}_{S_{p_2, \dots, p_k, p_1-3}^{(i-1)}}(t) \right].$$

□

Proof of Theorem 2.2. For simplicity of notation, we write u_j, v_j, w_j instead of $v_{1,j}, v_{2,j}, v_{3,j}$ respectively.

(a) Applying Proposition 3.1 to the splitting edge (v, u_1) of the tree $S_{p,q,r}^{(0)}$ we get

$$\text{cox}_{S_{p,q,r}^{(0)}}(t) = \text{cox}_{\mathbb{A}_p}(t) \text{cox}_{\mathbb{A}_{q+r+1}}(t) - t \text{cox}_{\mathbb{A}_{p-1}}(t) \text{cox}_{\mathbb{A}_q}(t) \text{cox}_{\mathbb{A}_r}(t).$$

The polynomial $\text{cox}_{\mathbb{A}_n}(t)$ can be easily calculated using Proposition 3.1. It satisfies the recurrence

$$\text{cox}_{\mathbb{A}_n}(t) = \text{cox}_{\mathbb{A}_{n-1}}(t) + t(\text{cox}_{\mathbb{A}_{n-1}}(t) - \text{cox}_{\mathbb{A}_{n-2}}(t))$$

and is given by the formula $\text{cox}_{\mathbb{A}_n}(t) = t^n + t^{n-1} + \dots + t + 1$.

Therefore

$$\begin{aligned} (t-1)^3 \text{cox}_{S_{p,q,r}^{(0)}}(t) &= t^{p+q+r+4} - 2t^{p+q+r+3} + t^{p+r+2} + t^{q+r+2} - t^{r+2} + \\ &\quad t^{p+q+2} - t^{p+2} - t^{q+2} + 2t - 1 \\ &= t^{p+q+r+2}(t-1) - t^{r+2}(t^q - 1) \text{cox}_{\mathbb{A}_{p-1}}(t) + \\ &\quad t^2(t^q - 1) \text{cox}_{\mathbb{A}_{p-1}}(t) - t + 1 \end{aligned}$$

and hence we get

$$\begin{aligned} (t-1) \text{cox}_{S_{p,q,r}^{(0)}}(t) &= t^{r+2} (t^{p+q} - \text{cox}_{\mathbb{A}_{p-1}}(t) \text{cox}_{\mathbb{A}_{q-1}}(t)) \\ &\quad + t^2 \text{cox}_{\mathbb{A}_{p-1}}(t) \text{cox}_{\mathbb{A}_{q-1}}(t) - 1 \\ &= t^{r+2} F_{p,q}^{(0)}(t) - (F_{p,q}^{(0)})^*(t). \end{aligned}$$

For the proof of $i = 1, 2$ we use the recurrence relation of Theorem 2.1. For $i = 1$, from Theorem 2.1 we get that

$$\begin{aligned}
\text{cox}_{S_{p,q,r}^{(1)}}(t) &= (t+1) \left[\text{cox}_{S_{p-1,q,r}^{(0)}}(t) - t \text{cox}_{S_{p-3,q,r}^{(0)}}(t) \right] \\
&= (t+1)t^{r+2} \left[F_{p-1,q}^{(0)}(t) - tF_{p-3,q}^{(0)}(t) \right] \\
&\quad - (t+1) \left[\left(F_{p-1,q}^{(0)} \right)^*(t) - t \left(F_{p-3,q}^{(0)} \right)^*(t) \right] \\
&= (t+1)t^{r+2} \left[F_{p-1,q}^{(0)}(t) - tF_{p-3,q}^{(0)}(t) \right] \\
&\quad - (t+1) \left[F_{p-1,q}^{(0)}(t) - tF_{p-3,q}^{(0)}(t) \right]^*.
\end{aligned}$$

The last equality holds because of the following fact. For $m_1 \geq m_2 \in \mathbb{N}$ and two polynomials f, g with degrees $\deg f = \deg(g) + m_1$ the reciprocal of the polynomial $f(t) + t^{m_2}g(t)$ is the polynomial $(f(t) + t^{m_2}g(t))^* = f^*(t) + t^{m_1-m_2}g^*(t)$. Therefore to finish the proof for the case $i = 1$ it is enough to show that

$$F_{p,q}^{(1)}(t) = F_{p-1,q}^{(0)}(t) - tF_{p-3,q}^{(0)}(t).$$

This is an easy verification:

$$\begin{aligned}
&F_{p-1,q}^{(0)}(t) - tF_{p-3,q}^{(0)}(t) = \\
&t^{p+q-2}(t-1) - \frac{t^{p-1}-1}{t-1} \text{cox}_{\mathbb{A}_{q-1}}(t) + t \frac{t^{p-3}-1}{t-1} \text{cox}_{\mathbb{A}_{q-1}}(t) = \\
&t^{p+q-2}(t-1) - (t^{p-2}+1) \text{cox}_{\mathbb{A}_{q-1}}(t).
\end{aligned}$$

For $i = 2$, by Theorem 2.1 we get

$$\begin{aligned}
\text{cox}_{S_{p,q,r}^{(2)}}(t) &= (t+1) \left[\text{cox}_{S_{q,p-1,r}^{(1)}}(t) - t \text{cox}_{S_{q,p-3,r}^{(1)}}(t) \right] \\
&= (t+1)t^{r+2} \left[F_{q,p-1}^{(1)}(t) - tF_{q,p-3}^{(1)}(t) \right] \\
&\quad - (t+1) \left[F_{q,p-1}^{(1)}(t) - tF_{q,p-3}^{(1)}(t) \right]^*
\end{aligned}$$

from which follows that to finish the proof for the case $i = 2$ is enough to verify that

$$F_{p,q}^{(2)}(t) = F_{q,p-1}^{(1)}(t) - tF_{q,p-3}^{(1)}(t).$$

(b) For the Coxeter polynomial $\text{cox}_{S_{p,q,r}^{(3)}}(t)$ we apply Proposition 3.1 to the edge (w_{r-2}, w_r) to obtain

$$\text{cox}_{S_{p,q,r}^{(3)}}(t) = (t+1) \text{cox}_{S_{p,q,r-1}^{(2)}}(t) - t(t+1) \text{cox}_{S_{p,q,r-3}^{(2)}}(t).$$

Therefore

$$\begin{aligned}
\frac{t-1}{(t+1)^3} \text{cox}_{S_{p,q,r}^{(3)}}(t) &= \frac{t-1}{(t+1)^2} \text{cox}_{S_{p,q,r-1}^{(2)}}(t) - t \frac{t-1}{(t+1)^2} \text{cox}_{S_{p,q,r-3}^{(2)}}(t) \\
&= t^{r+1} F_{p,q}^{(2)}(t) - \left(F_{p,q}^{(2)} \right)^*(t) - t^r F_{p,q}^{(2)}(t) + t \left(F_{p,q}^{(2)} \right)^*(t)
\end{aligned}$$

and hence we get

$$\text{cox}_{S_{p,q,r}^{(3)}}(t) = (t+1)^3 \left[t^r F_{p,q}^{(2)}(t) + (F_{p,q}^{(2)})^*(t) \right].$$

□

Remark 4.1. (a) For the case $i = 1$ we could have applied Proposition 3.1 to the splitting edge (u_{p-2}, u_p) and use that $S_{p,q,r}^{(0)} = S_{q,r,p}^{(0)}$ to obtain

$$\text{cox}_{S_{p,q,r}^{(1)}}(t) = (t+1) \left[t^p F_{q,r}^{(0)}(t) + (F_{q,r}^{(0)})^*(t) \right].$$

Similarly by noting that the graphs $S_{p,r,q}^{(1)}, S_{p,q,r}^{(1)}$ are the same and that the graphs $S_{p,q,r}^{(2)}, S_{q,p,r}^{(2)}$ are the same, Proposition 3.1 applied to the splitting edge (v_{q-2}, v_q) gives

$$\text{cox}_{S_{p,q,r}^{(2)}}(t) = (t+1)^2 \left[t^p F_{q,r}^{(1)}(t) + (F_{q,r}^{(1)})^*(t) \right].$$

(b) Explicitly the polynomials $F_{p,q}^{(i)}(t)$ are

$$\begin{aligned} F_{p,q}^{(0)}(t) &= \frac{t^p (t^{q+2} - 2t^{q+1} + 1) + t^q - 1}{(t-1)^2}, \\ F_{p,q}^{(1)}(t) &= \frac{t^{p-2} (t^{q+2} - 2t^{q+1} + 1) - t^q + 1}{t-1} \\ &= \frac{t^q (t^p - 2t^{p-1} - 1) + t^{p-2} - 1}{t-1}, \\ F_{p,q}^{(2)}(t) &= t^{p-2} (t^q - 2t^{q-1} - 1) - t^{q-2} - 1. \end{aligned}$$

Proof of Theorem 2.3 . (1) From Theorem 2.2 and 3.4 it is enough to show that the sequence $(\alpha_r)_{r \in \mathbb{N}}$ defined by $\alpha_r = \rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right)$, is convergent (note that from 2.5, $S_{p,q,r}^{(i)}$ are Salem trees and therefore $\alpha_r > 1$ for all $r \in \mathbb{N}$). It follows from 3.3 that for $i = 0, 1, 2$ the sequence $(\alpha_r)_{r \in \mathbb{N}}$ is increasing. Since the polynomial $\text{cox}_{S_{p,q,r}^{(i)}}(t)$ is written as $\text{cox}_{S_{p,q,r}^{(i)}}(t) = t^{r+2}F(t) + G(t)$ where $F(t), G(t)$ are monic polynomials, the sequence $(\alpha_r)_{r \in \mathbb{N}}$ is also bounded. For, if M is large enough such that the polynomials $F(t), G(t)$ are positive for all $t \geq M$, then $z < M$ for all $z \in Z \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right)$. Therefore the sequence $(\alpha_r)_{r \in \mathbb{N}}$ is indeed convergent.

We now prove that $\rho \left(F_{p,q}^{(i)}(t) \right)$ is a Pisot number (cf. Lemma 4.3 in [15]). Let $\epsilon > 0$ be small enough and r be large enough such that $\rho \left(\text{cox}_{S_{p,q,r}^{(i)}}(t) \right) > 1 + \epsilon$ and $\left| t^{r+2} F_{p,q}^{(i)}(t) \right| > \left| (F_{p,q}^{(i)})^*(t) \right|$ for every $|t| = 1 + \epsilon$. From Rouché's theorem (see [21]) it follows that the polynomial $F_{q,r}^{(i)}(t)$ has only one root, let us say z_0 , outside the unit circle. If z_0 was a Salem number then we would have $F^*(z_0) = 0$ and therefore $\text{cox}_{S_{p,q,r}^{(i)}}(z_0) = 0$ for all large r , contrary to 3.3. Therefore $z_0 = \rho \left(F_{p,q}^{(i)}(t) \right)$ and $\rho \left(F_{p,q}^{(i)}(t) \right)$ is a Pisot number.

(2) As in (1) we define the sequence $(\beta_p)_{p \in \mathbb{N}}$ by $\beta_p = \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t))$. Note that from 3.3, for $i = 1, 2, 3$, the sequence $(\beta_p)_{p \in \mathbb{N}}$ is decreasing. From 4.1 it follows that for $i = 1, 2$

$$(4.1) \quad \text{cox}_{S_{p,q,r}^{(i)}}(t) = (t+1)^i \left[t^p F_{q,r}^{(i-1)}(t) + (F_{q,r}^{(i-1)})^*(t) \right]$$

From Theorem 2.2 and from the fact that $\text{cox}_{S_{p,q,r}^{(3)}}(t) = \text{cox}_{S_{q,r,p}^{(3)}}(t)$ it follows that (4.1) holds for $i = 3$ also. Therefore the sequence $(\beta_p)_{p \in \mathbb{N}}$ is bounded and from 3.4 it converges to $\rho(F_{q,r}^{(i-1)}(t))$.

(3) For $q, r \in \mathbb{N}$ and $i \in \{0, 1, 2\}$ we define $\ell_{q,r}^{(i)} = \lim_{p \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t))$. By 3.3, $\ell_{q,r}$ is monotonic with respect to q . From (1) and (2) of this theorem and from the form of the polynomials $F_{q,r}^{(0)}(t), F_{q,r}^{(1)}(t)$, the sequence $(\ell_{q,r}^{(i)})_{q \in \mathbb{N}}$ is bounded and therefore convergent (note that $\ell_{q,r}^{(i)}$ equals $\rho(F_{q,r}^{(0)}(t))$ or $\rho(F_{q,r}^{(1)}(t))$). From 4.1, 3.4 and the fact that $\ell_{q,r} > 1$ we deduce that $\lim_{p,q \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t)) = \rho(t^{r+2} - 2t^{r+1} + 1)$.

(4) The proof for this case is similar to (3). For $p, q \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ we define $\ell_{p,q}^{(i)} = \lim_{r \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t))$. By 3.3, $\ell_{p,q}$ is monotonic with respect to q . From (1) and (2) of this theorem and from the form of the polynomials $F_{p,q}^{(1)}(t), F_{p,q}^{(2)}(t)$ (see 4.1), the sequence $(\ell_{p,q}^{(i)})_{q \in \mathbb{N}}$ is bounded and therefore convergent ($\ell_{p,q}^{(i)}$ is equal to $\rho(F_{p,q}^{(1)}(t))$ or $\rho(F_{p,q}^{(2)}(t))$). From 3.4 and the fact that $\ell_{p,q} > 1$ we deduce that $\lim_{q,r \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t)) = \rho(t^p - 2t^{p-1} - 1)$.

(5) Case $i = 0$ was proved by Lakatos in [14] and therefore we only consider the cases $i = 1, 2, 3$. Let $\ell_p^{(i)} = \lim_{q,r \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t))$. From (4), $\ell_p^{(i)} = \rho(H(t))$ where $H(t) := t^p - 2t^{p-1} - 1$. Hence $\lim_{p,q,r \rightarrow \infty} \rho(\text{cox}_{S_{p,q,r}^{(i)}}(t)) = \lim_{p \rightarrow \infty} \rho(H(t)) = 2$. \square

Proof of Theorem 2.4 . For $i \in \{0, 1, \dots, k-1\}$ we have

$$\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) = \frac{t^{p_k+1} F(t) - F^*(t)}{t-1}$$

where

$$F(t) = \text{cox}_{S_{p_1, \dots, p_{k-1}}^{(i)}}(t) - \text{cox}_{\mathbb{D}_{p_1}}(t) \dots \text{cox}_{\mathbb{D}_{p_i}}(t) \text{cox}_{\mathbb{A}_{p_{i+1}}}(t) \dots \text{cox}_{\mathbb{A}_{p_{k-1}}}(t).$$

Since the Coxeter polynomials of the trees $S_{p_1, \dots, p_k}^{(i)}$ and $\mathbb{D}_{p_j}, \mathbb{A}_{p_j}$ are self-reciprocal (see 1.2 (c)) the following relation holds

$$F^*(t) = \text{cox}_{S_{p_1, \dots, p_{k-1}}^{(i)}}(t) - t \text{cox}_{\mathbb{D}_{p_1}}(t) \dots \text{cox}_{\mathbb{D}_{p_i}}(t) \text{cox}_{\mathbb{A}_{p_{i+1}}}(t) \dots \text{cox}_{\mathbb{A}_{p_{k-1}}}(t).$$

Proposition 3.1 applied to the splitting edge $(v, v_{k,1})$ yields

$$\begin{aligned} \text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) &= \text{cox}_{S_{p_1, \dots, p_{k-1}}^{(i)}}(t) \text{cox}_{\mathbb{A}_{p_k}}(t) - \\ & t \text{cox}_{\mathbb{D}_{p_1}}(t) \dots \text{cox}_{\mathbb{D}_{p_i}}(t) \text{cox}_{\mathbb{A}_{p_{i+1}}}(t) \dots \text{cox}_{\mathbb{A}_{p_{k-1}}}(t) \text{cox}_{\mathbb{A}_{p_k-1}}(t) \\ &= \text{cox}_{S_{p_1, \dots, p_{k-1}}^{(i)}}(t) \frac{t^{p_k+1} - 1}{t - 1} - \\ & t \text{cox}_{\mathbb{D}_{p_1}}(t) \dots \text{cox}_{\mathbb{D}_{p_i}}(t) \text{cox}_{\mathbb{A}_{p_{i+1}}}(t) \dots \text{cox}_{\mathbb{A}_{p_{k-1}}}(t) \frac{t^{p_k} - 1}{t - 1} \end{aligned}$$

which is exactly the polynomial $\frac{t^{p_k+1}F(t) - F^*(t)}{t-1}$.

Therefore $\lim_{p_k \rightarrow \infty} \rho \left(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) \right) = \rho(F)$. Similar formulas hold for $i = k$ and inductively we show that

$$\lim_{p_2, \dots, p_k \rightarrow \infty} \rho \left(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) \right) = \rho(G)$$

where the polynomial $G(t)$ is given by

$$G(t) = \begin{cases} t^{p_1} - (k-1)t^{p_1-1} - k + 2, & \text{if } i \neq 0, \\ t^{p_1+1} - (k-1)t^{p_1} + k - 2, & \text{if } i = 0. \end{cases}$$

Hence

$$\lim_{p_1, p_2, \dots, p_k \rightarrow \infty} \rho \left(\text{cox}_{S_{p_1, \dots, p_k}^{(i)}}(t) \right) = k - 1.$$

□

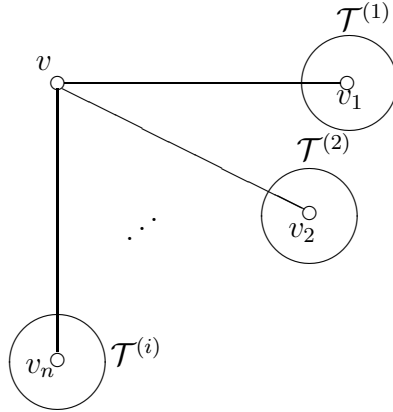


FIGURE 5. The join of the graphs $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(i)}$

Proof of Theorem 2.7. Let $\mathcal{T}^{(i)} = (\mathcal{T}_0^{(i)}, \mathcal{T}_1^{(i)})$ where $\mathcal{T}_0^{(i)}$ is the set of the vertices of $\mathcal{T}^{(i)}$. We denote by $\mathcal{T}^{[i]}$ the join of the graphs $\mathcal{T}^{(1)}, \dots, \mathcal{T}^{(i)}$ at the vertices $v_i \in \mathcal{T}_0^{(i)}$. The graph $\mathcal{T}^{(i)}$ looks like the one in Figure 5.

Let $i \in \{2, 3, \dots, k\}$. Applying Proposition 3.1 to the edge (v, v_i) we get

$$\text{cox}_{\mathcal{T}^{[i]}}(t) = \text{cox}_{\mathcal{T}^{[i-1]}}(t) \text{cox}_{\mathcal{T}^{(i)}}(t) - t \text{cox}_{\mathcal{T}^{(1)}}(t) \dots \text{cox}_{\mathcal{T}^{(i-1)}}(t) \text{cox}_{\widetilde{\mathcal{T}^{(i)}}}(t),$$

where we denote by $\widetilde{\mathcal{T}^{(i)}}$ the induced subgraph of $\mathcal{T}^{(i)}$ with the set of vertices $\widetilde{\mathcal{T}^{(i)}}_0 = \mathcal{T}_0^{(i)} \setminus \{v_i\}$.

Let us write $P_k(t) = \text{cox}_{\mathcal{T}(1)}(t) \dots \text{cox}_{\widetilde{\mathcal{T}(i)}}(t) \dots \text{cox}_{\mathcal{T}(k)}(t)$. Then we have

$$\begin{aligned}
\text{cox}_{\mathcal{T}[k]}(t) &= \text{cox}_{\mathcal{T}[k-1]}(t) \text{cox}_{\mathcal{T}(k)}(t) - tP_k(t) \\
&= \text{cox}_{\mathcal{T}[k-2]}(t) \text{cox}_{\mathcal{T}(k-1)}(t) \text{cox}_{\mathcal{T}(k)}(t) - \\
t \text{cox}_{\mathcal{T}(1)}(t) \dots \text{cox}_{\mathcal{T}(k-2)}(t) \text{cox}_{\widetilde{\mathcal{T}(k-1)}}(t) \text{cox}_{\mathcal{T}(k)}(t) &- tP_k(t) \\
&= \text{cox}_{\mathcal{T}[k-2]}(t) \text{cox}_{\mathcal{T}(k-1)}(t) \text{cox}_{\mathcal{T}(k)}(t) - t(P_{k-1}(t) + P_k(t)) \\
&\dots \\
&= \text{cox}_{\mathcal{T}[0]}(t) \text{cox}_{\mathcal{T}(1)}(t) \dots \text{cox}_{\mathcal{T}(k)}(t) - t(P_1(t) + \dots + P_k(t)) \\
&= (t+1) \text{cox}_{\mathcal{T}(1)}(t) \dots \text{cox}_{\mathcal{T}(k)}(t) - t(P_1(t) + \dots + P_k(t)).
\end{aligned}$$

Since z is a root of the polynomial $P_i(t)$ of multiplicity $m - m_i$, the theorem follows. \square

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